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# Confluent polynomial Vandermonde-like matrices: displacement structures, inversion formulas and fast algorithm<sup>☆</sup>

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## Abstract

In the present paper we use the displacement structure approach to introduce a new class of what are called confluent polynomial Vandermonde-like matrices, which generalize the ordinary simple nodes polynomial Vandermonde matrices studied earlier by different authors. The displacement structure approach leads to the explicit inversion formulas for confluent polynomial Vandermonde-like matrices and fast algorithms for inversion and for solving the associated linear systems.

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## 1. Introduction

Undoubtedly, the Vandermonde matrix is one of the most important matrices in applied matrix theory. In general, the most interesting properties of Vandermonde

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matrices are their simple displacement structures, their relations with Toeplitz, Hankel and Cauchy matrices and connections with interpolation problems. For any given nodes sequence  $x = (x_1, x_2, \dots, x_n)$  with  $x_i$  pairwise distinct and any polynomials sequence  $Q(x) = \{Q_0(x), Q_1(x), \dots, Q_{n-1}(x)\}$  with  $\deg Q_k(x) = k$ , we may associate a so-called polynomial Vandermonde matrix  $V_Q(x) = [Q_j(x_i)]_{i=1, j=0}^{n, n-1}$  [2]. When  $Q(x)$  stands for the standard power basis  $P = \{1, x, \dots, x^{n-1}\}$ , Chebyshev polynomials bases  $U$  and  $T$  of the second and first kind, respectively, we get the classical Vandermonde and Chebyshev–Vandermonde matrices  $V_P(x)$ ,  $V_U(x)$  and  $V_T(x)$ , respectively (see, e.g., [3,12]). The displacement can be traced back to the early work on solving linear systems by Schur [15]. The concept of displacement structure was first introduced in [5] (see also [6–8]) using the Stein-type displacement operator  $\nabla_{\{F,A\}}(\cdot)$  given by

$$\nabla_{\{F,A\}}(R) = R - FRA. \quad (1.1)$$

In general, the generalized displacement equation is defined by

$$\nabla_{\{\Omega, A, F, A\}}(R) = \Omega R A - F R A \quad (1.2)$$

(see, e.g., [2,4,10]), if  $\nabla_{\{\Omega, A, F, A\}}(R)$  has low rank, say  $\alpha$ , independent of  $n$ , then  $R$  is said to be *structured* or a *displacement structure* matrix with respect to the displacement operator defined by (1.2), and  $\alpha$  is referred to as the *displacement rank* of  $R$ . A special case of (1.2) will have a more simple Sylvester form

$$\nabla_{\{\Omega, I, I, A\}}(R) = \Omega R - R A \quad (1.3)$$

which was first studied by Heinig [16] for Cauchy-like and Vandermonde-like matrices (see also [2,9]).

In the sense of (1.2), polynomial Vandermonde matrix  $V_Q(x)$  possesses three kinds of displacement structures with displacement rank 1 (see [2, Lemmas 1.1–1.2 and 1.4]), therefore, *polynomial Vandermonde-like* matrices are defined as matrices with low displacement rank. In [2], inversion formulas, Gaussian elimination with partial pivoting and fast inverse algorithm for polynomial Vandermonde-like matrices were given. All results were derived under the condition that the nodes sequence  $x$  is simple, i.e., all  $x_i$  pairwise distinct. We point out that this condition is special and unnecessary since in many application situations, for example, in interpolation problem, the interpolation nodes  $x_i$  are usually multiple.

In this paper we generalize all the above results to the multiple nodes case—*confluent polynomial Vandermonde-like* matrices, which can be seen the generalization and supplement to paper [2]. The paper is organized as follows. In Section 2 we introduce three displacement operators associated with confluent polynomial Vandermonde matrices and define the class of confluent polynomial Vandermonde-like matrices. In Section 3 we list some necessary properties for generalized Horner polynomials and for change of basis. In Section 4 we derive two inversion formulas for confluent polynomial Vandermonde-like matrices, and as special case, we give the generalizations of inversion formulas for simple Chebyshev Vandermonde-like

matrices [3,12]. Section 5 is devoted to the implementation of block Gaussian elimination for solving associated linear system and fast inverse algorithm. In Sections 6 and 7 we point out the relationship of confluent polynomial Vandermonde-like matrices with confluent Cauchy-like matrices and  $q$ -adic Vandermonde matrices over any non-algebraically closed field  $\mathbf{F}$ .

## 2. Confluent polynomial Vandermonde-like matrices

In this section, we give three displacement operators for the confluent polynomial Vandermonde matrix and define the class of confluent polynomial Vandermonde-like matrices. Following [2], let  $Q(x) = \{Q_0(x), Q_1(x), \dots, Q_n(x)\}$  be a system of  $n + 1$  polynomials satisfying the recurrence relations

$$\begin{aligned} Q_0(x) &= \alpha_0, \\ Q_k(x) &= \alpha_k x Q_{k-1}(x) - a_{k-1,k} Q_{k-1}(x) - a_{k-2,k} Q_{k-2}(x) \\ &\quad - \dots - a_{0,k} Q_0(x), \end{aligned} \quad (2.1)$$

where  $\alpha_k \neq 0$ . Also let

$$x = (\underbrace{x_1, \dots, x_1}_{n_1}, \underbrace{x_2, \dots, x_2}_{n_2}, \dots, \underbrace{x_t, \dots, x_t}_{n_t}) \quad (2.2)$$

be a sequence of nodes, with  $x_i$ 's pairwise distinct and  $\sum_{i=1}^t n_i = n$ , and let

$$J_x = \text{diag}(J_{x_i})_{i=1}^t, \quad \text{where } J_{x_i} = \begin{bmatrix} x_i & 0 & \dots & 0 \\ 1 & x_i & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & x_i \end{bmatrix} \in C^{n_i \times n_i}, \quad (2.3)$$

be the Jordan canonical form corresponding to  $x$ .

In connection with polynomial system  $Q(x) = \{Q_0(x), \dots, Q_{n-1}(x)\}$  and nodes sequence  $x$  we define the *confluent polynomial Vandermonde* matrix as follows:

$$V_Q(x) = \begin{bmatrix} V_Q(x_1) \\ V_Q(x_2) \\ \vdots \\ V_Q(x_t) \end{bmatrix}, \quad (2.4)$$

where

$$V_Q(x_i) = \left[ \frac{1}{j!} \frac{d^{(j)} Q_k(x_i)}{dx^j} \right]_{j=0, k=0}^{n_i-1, n-1} = \begin{bmatrix} Q(x_i) \\ \frac{Q'(x_i)}{1!} \\ \vdots \\ \frac{Q^{(n_i-1)}(x_i)}{(n_i-1)!} \end{bmatrix}$$

$$= \text{col} \left( \frac{1}{j!} \frac{d^j Q(x_i)}{dx^j} \right)_{j=0}^{n_i-1}.$$

### 2.1. First displacement operator

From the coefficients  $\alpha_k$  and  $a_{i,k}$  in (2.1) we introduce the matrices (see, e.g., [2,20])

$$M_Q = \begin{bmatrix} 1 & a_{01} & a_{02} & \cdots & a_{0,n-1} \\ 0 & 1 & a_{12} & \cdots & a_{1,n-1} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad (2.5)$$

$$N_Q = \begin{bmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & & & \ddots & \alpha_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

Then the following statement holds:

**Lemma 2.1.** *Let the polynomials  $Q = \{Q_0(x), \dots, Q_n(x)\}$  be defined by (2.1), the matrices  $M_Q$  and  $N_Q$  be given by (2.5), and  $J_x$  be given by (2.3). Then the confluent polynomial Vandermonde matrix  $V_Q(x)$  given by (2.4) satisfies*

$$\begin{aligned} \nabla_{\{I, M_Q, J_x, N_Q\}}(V_Q(x)) &= V_Q(x)M_Q - J_x V_Q(x)N_Q \\ &= \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_t \end{bmatrix} [\alpha_0 \quad 0 \quad \cdots \quad 0], \end{aligned} \quad (2.6)$$

where  $e_i = [1 \ 0 \ \cdots \ 0]^T \in C^{n_i \times 1}$  is a unit column vector of length  $n_i$ .

**Proof.** Rewriting the recurrence relations (2.1) as matrix form, we have

$$Q(x)M_Q - x \cdot Q(x)N_Q = [\alpha_0 \quad 0 \quad \cdots \quad 0] \quad (2.7)$$

here  $Q(x) = (Q_0(x), Q_1(x), \dots, Q_{n-1}(x))$ , then taking  $j$ th derivative at  $x_i$  and dividing by  $j!$  in both sides of (2.7) ( $i = 1, \dots, t, j = 0, \dots, n_i - 1$ ), the (2.6) is obtained.  $\square$

## 2.2. Second displacement operator

Following [2] and [20], define for a polynomial

$$\Theta(x) = \theta_0 Q_0(x) + \theta_1 Q_1(x) + \cdots + \theta_{n-1} Q_{n-1}(x) + \theta_n Q_n(x) \quad (\theta_n \neq 0)$$

its confederate matrix

$$C_Q(\Theta) = \begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{\theta_0}{\theta_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{\theta_1}{\theta_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{\theta_2}{\theta_n} \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{\theta_{n-1}}{\theta_n} \end{bmatrix} \quad (2.8)$$

with respect to the system  $Q$ . In the simplest case of the standard power basis  $P = \{1, x, \dots, x^n\}$ ,  $C_P(\Theta)$  reduces to the well-known companion matrix

$$C_P(\Theta) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{\theta_0}{\theta_n} \\ 1 & 0 & \cdots & 0 & -\frac{\theta_1}{\theta_n} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{\theta_{n-1}}{\theta_n} \end{bmatrix}.$$

**Lemma 2.2.** Let polynomials system  $Q = \{Q_0(x), \dots, Q_n(x)\}$  be defined by (2.1), let the polynomial  $\Theta(x)$  be arbitrary, and let  $C_Q(\Theta)$  be confederate matrix as in (2.8) of  $\Theta(x)$ . Then

$$\begin{aligned} \nabla_{\{J_x, I, I, C_Q(\Theta)\}}(V_Q(x)) &= J_x V_Q(x) - V_Q(x) C_Q(\Theta) \\ &= \begin{bmatrix} \Theta(x_1) \\ \vdots \\ \frac{\Theta^{(n_1-1)}(x_1)}{(n_1-1)!} \\ \vdots \\ \Theta(x_t) \\ \vdots \\ \frac{\Theta^{(n_t-1)}(x_t)}{(n_t-1)!} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\alpha_n \theta_n} \end{bmatrix}. \end{aligned} \quad (2.9)$$

In particular, if  $x_1, x_2, \dots, x_t$  are the zeros with multiplicities  $n_1, n_2, \dots, n_t$  of polynomial  $\Theta(x)$ , thus,  $\Theta(x) = \theta_n \prod_{i=1}^t (x - x_i)^{n_i}$ , then

$$V_Q(x) C_Q(\Theta) V_Q(x)^{-1} = J_x. \quad (2.10)$$

**Proof.** Note that  $Q_n(x) = \frac{\Theta(x)}{\theta_n} - \frac{\theta_0}{\theta_n} Q_0(x) - \cdots - \frac{\theta_{n-1}}{\theta_n} Q_{n-1}(x)$ . Rewriting the recurrence relations (2.1) as matrix form, we have

$$x \cdot Q(x) - Q(x)C_Q(\Theta) = \Theta(x) \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\alpha_n \theta_n} \end{bmatrix} \quad (2.11)$$

here  $Q(x) = (Q_0(x), \dots, Q_{n-1}(x))$ , then taking  $j$ th derivatives at  $x_i$  and dividing by  $j!$  in both sides of (2.11) ( $i = 1, \dots, t, j = 0, 1, \dots, n_i - 1$ ), the (2.9) is obtained.  $\square$

### 2.3. Third displacement operator

Following [2], we introduce an upper and triangular matrix of the form

$$W_Q = \begin{bmatrix} 0 & w_{12} & w_{13} & \cdots & w_{1n} \\ 0 & 0 & w_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & w_{n-2,n} \\ \vdots & & & \ddots & w_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \quad (2.12)$$

whose entries are specified by

$$\begin{aligned} Q_1(x) &= \delta_1 Q_0(x) + x w_{12} Q_0(x), \\ Q_k(x) &= \delta_k Q_0(x) + x \cdot \sum_{i=1}^k w_{i,k+1} Q_{i-1}(x) \quad (k = 2, 3, \dots, n-1). \end{aligned} \quad (2.13)$$

Note that since the polynomials  $\{Q_0(x), x Q_0(x), \dots, x Q_{n-1}(x)\}$  form a basis in the linear space  $C_n[x]$  of all polynomials of degree not exceeding  $n$ , the numbers  $\delta_k$  and  $w_{ij}$  are uniquely determined by (2.13).

**Lemma 2.3.** Let  $x_1, x_2, \dots, x_t$  be  $t$  non-zero numbers,

$$\Omega = J_{1/x} = J_x^{-1} = \text{diag}(J_{1/x_1}, \dots, J_{1/x_t}) \quad (2.14)$$

and let the matrix  $W_Q$  be defined as in (2.12). Then

$$\begin{aligned} \nabla_{\{J_{1/x}, I, I, W_Q\}}(V_Q(x)) &= J_{1/x} V_Q(x) - V_Q(x) W_Q \\ &= \begin{bmatrix} J_{1/x_1} e_1 \\ J_{1/x_2} e_2 \\ \vdots \\ J_{1/x_t} e_t \end{bmatrix} [\alpha_0 \quad \alpha_0 \delta_1 \quad \cdots \quad \alpha_0 \delta_{n-1}], \end{aligned} \quad (2.15)$$

where  $e_i$  are defined as in Lemma 2.1.

**Proof.** Rewriting the recurrence relations (2.15) as matrix form, we have

$$\frac{1}{x}Q(x) - Q(x)W_Q = \frac{1}{x} \begin{bmatrix} \alpha_0 & \alpha_0\delta_1 & \cdots & \alpha_0\delta_{n-1} \end{bmatrix} \quad (2.16)$$

here  $Q(x) = (Q_0(x), Q_1(x), \dots, Q_{n-1}(x))$ , then taking  $j$ th derivatives at  $x_i$  and dividing by  $j!$  in both sides of (2.16) ( $i = 1, \dots, t, j = 0, 1, \dots, n_i - 1$ ), then (2.15) is obtained.  $\square$

#### 2.4. Examples

In this part we give several examples where the displacement operators (2.6), (2.9) and (2.15) are specialized to yield the ordinary confluent Vandermonde and confluent Chebyshev–Vandermonde matrices.

**Example 1.** Let  $P$  stand for the standard power basis  $P = \{1, x, \dots, x^{n-1}\}$ , i.e.,  $\alpha_k = 1, a_{ik} = 0$  for all  $i = 0, 1, \dots, n-1, k = 1, 2, \dots, n$ . For the confluent Vandermonde matrix  $V_P(x)$  the displacement equations (2.6), (2.9) and (2.15), respectively, have the forms

$$V_P(x) - J_x V_P(x) Z_0 = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_t \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$J_x V_P(x) - V_P(x) C_P(\Theta) = \begin{bmatrix} \Theta(x_1) \\ \vdots \\ \frac{\Theta^{(n_1-1)}(x_1)}{(n_1-1)!} \\ \vdots \\ \Theta(x_t) \\ \vdots \\ \frac{\Theta^{(n_t-1)}(x_t)}{(n_t-1)!} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\theta_n} \end{bmatrix},$$

where  $C_P(\Theta)$  is the companion matrix of  $\Theta(x)$  and  $Z_0$  is the backforward shift matrix, and

$$J_{1/x} V_P(x) - V_P(x) Z_0 = \begin{bmatrix} J_{1/x_1} e_1 \\ J_{1/x_2} e_2 \\ \vdots \\ J_{1/x_t} e_t \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \quad (2.17)$$

**Example 2.** Let  $U$  and  $T$  stand for the bases  $U = \{U_0(x), U_1(x), \dots, U_{n-1}(x)\}$  and  $T = \{T_0(x), T_1(x), \dots, T_{n-1}(x)\}$  of Chebyshev polynomials of the second and first kind, respectively, i.e., the recurrence relations (2.1) reduces

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_k(x) = 2xU_{k-1}(x) - U_{k-2}(x),$$

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x).$$

For the confluent Chebyshev–Vandermonde matrices  $V_U(x)$  and  $V_T(x)$ , Eq. (2.15) has the forms, respectively,

$$\begin{aligned} J_{1/x} V_U(x) - V_U(x) \cdot 2 \sum_{i=1}^{[n/2]} (-1)^{i-1} Z_0^{2i-1} \\ = \begin{bmatrix} J_{1/x_1} e_1 \\ \vdots \\ J_{1/x_t} e_t \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \end{bmatrix} \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} J_{1/x} (V_T(x) D_0) - (V_T(x) D_0) \cdot 2 \sum_{i=1}^{[n/2]} (-1)^{i-1} Z_0^{2i-1} \\ = \begin{bmatrix} J_{1/x_1} e_1 \\ \vdots \\ J_{1/x_t} e_t \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \end{bmatrix}, \end{aligned} \quad (2.19)$$

where  $D_0 = \text{diag}(\frac{1}{2} \ 1 \ \cdots \ 1)$ .

Lemmas 2.1–2.3 claim that a confluent polynomial Vandermonde matrix  $V_Q(x)$  is transformed to a rank one matrix by each of the displacement operators in (2.6), (2.9), and (2.15). Following [2], any of these displacement operators can be chosen for defining a more general class of matrices with low displacement rank, the following theorems imply that if a matrix has low displacement rank with respect to one of the above displacement operators, then it has low displacement rank with respect to the other two. Therefore, all three definitions describe the same class of matrices, which we shall call the class of *confluent polynomial Vandermonde-like* matrices. The proofs can refer to [2, Theorems 2.1–2.3].

**Theorem 2.4.** Let  $\nabla_{\{I, M_Q, J_x, N_Q\}}$  and  $\nabla_{\{J_x, I, I, C_Q(\theta)\}}$  be the first and second displacement operators given by (2.6) and (2.9), respectively. Then for any matrix  $R \in C^{n \times n}$ , we have

$$|\text{rank } \nabla_{\{I, M_Q, J_x, N_Q\}}(R) - \text{rank } \nabla_{\{J_x, I, I, C_Q(\theta)\}}(R)| \leq 2.$$



**Theorem 2.5.** Let  $\nabla_{\{I, M_Q, J_x, N_Q\}}$  and  $\nabla_{\{J_{1/x}, I, I, W_Q\}}$  be the first and third displacement operators defined by (2.6) and (2.15), respectively. Then  $W_Q = N_Q M_Q^{-1}$ , and for any matrix  $R \in C^{n \times n}$ , we have

$$\text{rank } \nabla_{\{I, M_Q, J_x, N_Q\}}(R) = \text{rank } \nabla_{\{J_{1/x}, I, I, W_Q\}}(R).$$

### 3. Generalized Horner polynomials and change of basis

The most of this section can be found in [2,20]. We only list the main and necessary results, for the corresponding proofs one may refer to [2,20].

Association with a given system  $Q = \{Q_0(x), Q_1(x), \dots, Q_n(x)\}$ , the generalized Horner polynomials system  $\widehat{Q} = \{\widehat{Q}_0(x), \widehat{Q}_1(x), \dots, \widehat{Q}_n(x)\}$  are defined by

$$\widehat{Q}_n(x) = Q_n(x) \quad \text{and} \quad \frac{Q_n(x) - Q_n(y)}{x - y} = \sum_{i=0}^{n-1} Q_i(x) \widehat{Q}_{n-1-i}(y).$$

Let  $\widehat{Q}$  has the same form of recurrence relations as  $Q$  in (2.1), thus

$$\begin{aligned} \widehat{Q}_0(x) &= \hat{\alpha}_0, \\ \widehat{Q}_k(x) &= \hat{\alpha}_k x \widehat{Q}_{k-1}(x) - \hat{a}_{k-1,k} \widehat{Q}_{k-1}(x) - \hat{a}_{k-2,k} \widehat{Q}_{k-2}(x) \\ &\quad - \dots - \hat{a}_{0,k} \widehat{Q}_0(x). \end{aligned} \quad (3.1)$$

Then we have the following relations:

$$\hat{\alpha}_k = \alpha_{n-k}, \quad \hat{a}_{k,j} = \frac{\alpha_{n-j}}{\alpha_{n-k}} a_{n-j,n-k} \quad (3.2)$$

( $k = 0, 1, \dots, n-1, j = 1, 2, \dots, n$ ). These relations can be reformed in matrix form as following

**Lemma 3.1.** Let  $Q_k(x)$  and  $\widehat{Q}_k(x)$  be the systems of polynomials specified by (2.1) and (3.1), respectively. Then the following statements hold:

$$\begin{aligned} \text{(i)} \quad Q_n(x) &= \widehat{Q}_n(x), \\ \text{(ii)} \quad C_{\widehat{Q}}(\widehat{Q}_n) &= \widetilde{I} \cdot C_Q(Q_n)^T \cdot \widetilde{I}, \end{aligned} \quad (3.3)$$

where  $\widetilde{I}$  stands for the antiidentity matrix.

$$\text{(iii)} \quad W_{\widehat{Q}} = \widetilde{I} \cdot W_Q^T \cdot \widetilde{I}, \quad (3.4)$$

where  $W_Q$  and  $W_{\widehat{Q}}$  are the matrices defined by (2.12) and (2.13).

Note that when  $Q$  stands for the power basis  $P$ , according to formula (3.2) or (3.3), we have

$$\widehat{P} = \{1, x, \dots, x^{n-1}\} = P. \quad (3.5)$$

Similarly, when  $Q$  stands for the Chebyshev polynomials bases  $U$  and  $T$  of the second and first kind, we have

$$\widehat{U} = \{U_0(x), U_1(x), \dots, U_{n-1}(x)\} = U, \quad (3.6)$$

$$\widehat{T} = \{U_0(x), U_1(x), \dots, U_{n-2}(x), \frac{1}{2}U_{n-1}(x)\}. \quad (3.7)$$

Along with the polynomials  $Q = \{Q_1(x), \dots, Q_n(x)\}$ , consider another set of polynomials  $R = \{R_0(x), \dots, R_n(x)\}$  defined by analogous recurrences as (2.1), and let

$$S_{RQ} = [S_{ij}]_{i,j=1}^n \quad (3.8)$$

be the upper triangular matrix corresponding to passing from the basis  $R$  to the basis  $Q$  in the linear space  $C_n[x]$  of all polynomials with degree not exceeding  $n$ , then we have

$$V_Q(x) = V_R(x) \cdot S_{RQ}. \quad (3.9)$$

Furthermore, we have

**Lemma 3.2.** *Let  $S_{RQ}$  be defined by (3.8). Then*

$$(i) C_Q(\Theta) = S_{RQ}^{-1} C_R(\Theta) S_{RQ}, \quad (3.10)$$

$$(ii) W_Q = S_{RQ}^{-1} W_R S_{RQ}. \quad (3.11)$$

The columns  $s_k \in C^n$  of the matrix  $S_{RQ} = [s_1 \ s_2 \ \dots \ s_n]$  can be computed recursively:

$$(iii) \ s_1 = [\alpha_0/\beta_0 \ 0 \ \dots \ 0]^T \text{ and} \\ s_{k+1} = \alpha_k C_R(\Theta) s_k - a_{k-1,k} s_{k-1} - a_{k-2,k} s_{k-2} - \dots - a_{0,k} s_1. \quad (3.12)$$

For the simplest case where  $R$  stands for the power basis  $P = \{1, x, \dots, x^{n-1}\}$ , we have

$$(iv) \ s_1 = [\alpha_0 \ 0 \ \dots \ 0]^T \text{ and} \\ s_{k+1} = \alpha_k Z_0 s_k - a_{k-1,k} s_{k-1} - \dots - a_{0,k} s_1. \quad (3.13)$$

Also in the simplest case the first row of the matrix  $S_{PQ}$  is given by

$$[s_{1k}] = [\alpha_0 \ \alpha_0 \delta_1 \ \dots \ \alpha_0 \delta_{n-1}], \quad (3.14)$$

where  $\delta_k$  are as in (2.13).

$$(v) \ W_Q = S_{PQ}^{-1} Z_0 S_{PQ}. \quad (3.15)$$

#### 4. Inversion formulas for confluent polynomial Vandermonde-like matrices

In this section we generalize the displacement equations and inversion formulas for the simple polynomial Vandermonde-like matrices in [2] to the confluent (multiple nodes) case. To this end, we introduce some notations. For an arbitrary column vector  $a \in C^n$ , let  $a$  be partitioned into  $t$  subcolumns in compatiency with nodes vector  $x$ , thus  $a = \text{col}(a_i)_{i=1}^t$ ,  $a_i \in C^{n_i \times 1}$ . Here and henceforth, by  $L(a) = \text{diag}(L(a_i))_{i=1}^t$  is denoted a block diagonal matrix, where  $L(a_i)$  is denoted by a lower triangular Toeplitz matrix with first column  $a_i \in C^{n_i}$ . Also let  $J = \text{diag}(J_i)_{i=1}^t$ , where  $J_i$  is  $n_i \times n_i$  antiidentity matrix. Note that  $J$  is different from  $\tilde{J}$ , and  $J^T = J$ ,  $J^2 = J$ .

##### 4.1. First inversion formula

The following lemma shows how any matrix can be recovered from its  $\{J_{1/x}, I, I, W_Q\}$  displacement.

**Lemma 4.1.** *Let  $Q$  be the system of polynomials specified by (2.1). Then for given matrices*

$$G = [g^{(1)}, \dots, g^{(\alpha)}] = \text{row}(g^{(k)})_{k=1}^\alpha, \quad g^{(k)} = \text{col}[\text{col}(g_{ij}^{(k)})_{j=0}^{n_i-1}]_{i=1}^t \in C^{n \times 1},$$

$$B = \text{col}(b_k)_{k=1}^\alpha, \quad b_k = \text{row}(b_{kj})_{j=1}^n \in C^{1 \times n}, \quad (4.0)$$

the unique solution  $R \in C^{n \times n}$  of the equation

$$\nabla_{\{J_{1/x}, I, I, W_Q\}}(R) = J_{1/x}R - RW_Q = GB \quad (4.1)$$

is given by

$$R = \sum_{k=1}^\alpha L(c^{(k)}) \cdot V_Q(x) \cdot \left( \sum_{j=1}^n d_{kj} W_Q^{j-1} \right), \quad (4.2)$$

where

$$c^{(k)} = J_x g^{(k)} \in C^n \quad \text{or} \quad \begin{bmatrix} c^{(1)} & \dots & c^{(\alpha)} \end{bmatrix} = J_x G$$

and

$$d_k = [d_{k1} \quad d_{k2} \quad \dots \quad d_{kn}] = b_k S_{QP},$$

where  $P = \{1, x, \dots, x^{n-1}\}$  and  $S_{QP}$  is defined as in (3.8).

**Proof.** First note that since the spectra of the matrices  $J_{1/x}$  and  $W_Q$  have no intersection, there is only one solution of Eq. (4.1) (see, for example, [21, p. 414]). Substituting  $R$  given by (4.2) into (4.1) and then using (2.15), we have

$$\begin{aligned}
\nabla_{\{J_{1/x}, I, I, W_Q\}}(R) &= \sum_{k=1}^{\alpha} L(c^{(k)}) \cdot [J_{1/x} V_Q(x) - V_Q(x) W_Q] \cdot \sum_{j=1}^n d_{kj} W_Q^{j-1} \\
&= \sum_{k=1}^{\alpha} L(c^{(k)}) \cdot \begin{bmatrix} J_{1/x_1} e_1 \\ \vdots \\ J_{1/x_t} e_t \end{bmatrix} [\alpha_0 \quad \alpha_0 \delta_1 \quad \cdots \quad \alpha_0 \delta_{n-1}] \\
&\quad \times \sum_{j=1}^n d_{kj} W_Q^{j-1}.
\end{aligned}$$

Furthermore, since lower triangular Toeplitz matrices commute, we have  $L(c_i^{(k)}) \cdot J_{1/x_i} e_i = L(J_{x_i} g_i^{(k)}) \cdot J_{1/x_i} e_i = J_{1/x_i} \cdot L(J_{x_i} g_i^{(k)}) \cdot e_i = L(J_{1/x_i} \cdot J_{x_i} g_i^{(k)}) \cdot e_i = g_i^{(k)}$ . This fact, using (3.15) and that the first row of  $S_{PQ}$  is given by (3.14), we have

$$\begin{aligned}
\nabla_{\{J_{1/x}, I, I, W_Q\}}(R) &= \sum_{k=1}^{\alpha} L(c^{(k)}) \cdot \begin{bmatrix} J_{1/x_1} e_1 \\ \vdots \\ J_{1/x_t} e_t \end{bmatrix} [\alpha_0 \quad \alpha_0 \delta_1 \quad \cdots \quad \alpha_0 \delta_{n-1}] \\
&\quad \times S_{PQ}^{-1} \left( \sum_{j=1}^n d_{kj} Z_0^{j-1} \right) \cdot S_{PQ} \\
&= \sum_{k=1}^{\alpha} \text{col}(g_i^{(k)})_{i=1}^t [1 \quad 0 \quad \cdots \quad 0] \cdot \left( \sum_{j=1}^n d_{kj} Z_0^{j-1} \right) \cdot S_{PQ} \\
&= \sum_{k=1}^{\alpha} g^{(k)} \cdot d_k \cdot S_{PQ} = \sum_{k=1}^{\alpha} g^{(k)} \cdot b_k = GB
\end{aligned}$$

and (4.2) follows.  $\square$

**Theorem 4.2.** Let  $Q$  be a system of polynomials given by (2.1). Then

$$\text{rank } \nabla_{\{J_{1/x}, I, I, W_Q\}}(R) = \text{rank } \nabla_{\{J_{1/x}, I, I, W_{\widehat{Q}}\}}(JR^{-T} \widetilde{I}), \quad (4.3)$$

where  $\widehat{Q}$  is the associated system specified by (3.1) and (3.2). Moreover, if  $R$  is specified by its generator  $\{G, B\}$  on the right-hand side of

$$\nabla_{\{J_{1/x}, I, I, W_Q\}}(R) = J_{1/x} R - RW_Q = GB. \quad (4.4)$$

Then

$$R^{-1} = \widetilde{I} \cdot \sum_{k=1}^{\alpha} \left( \sum_{j=1}^n d_{kj} (W_Q^T)^{j-1} \right) \cdot V_Q^T(x) \cdot L(c^{(k)}), \quad (4.5)$$

where  $c^{(k)}$  and  $d_{kj}$  are determined from  $2\alpha$  linear system of equations

$$\begin{aligned} [c^{(1)}, \dots, c^{(\alpha)}] &= J_x \cdot J R^{-T} B^T \in C^{n \times \alpha}, \\ \begin{bmatrix} d_1 \\ \vdots \\ d_\alpha \end{bmatrix} &= [d_{kj}]_{k,j=1}^{\alpha,n} = G^T R^{-T} \tilde{I} S_{\hat{Q}P} \in C^{\alpha \times n}. \end{aligned} \quad (4.6)$$

**Proof.** Multiplying (4.4) by  $\tilde{I}R^{-1}$  from the left and by  $R^{-1}$  from the right, then taking transpose, we obtain

$$J_{1/x}^T (R^{-T} \tilde{I}) - (R^{-T} \tilde{I}) W_{\hat{Q}} = (R^{-T} B^T) (G^T R^{-T} \tilde{I}) \quad (4.7)$$

where we have used (3.4). Note that  $J_{1/x}^T = J \cdot J_{1/x} \cdot J$ , then (4.7) is equivalent to the following equality:

$$J_{1/x} (J R^{-T} \tilde{I}) - (J R^{-T} \tilde{I}) W_{\hat{Q}} = (J R^{-T} B^T) (G^T R^{-T} \tilde{I}). \quad (4.8)$$

Eq. (4.8) means that the  $\{J_{1/x}, I, I, W_{\hat{Q}}\}$ -displacement rank of the matrix  $J R^{-T} \tilde{I}$  is equal to the  $\{J_{1/x}, I, I, W_Q\}$ -displacement rank of matrix  $R$ , which proves (4.3). Writing then (4.2) for the matrix  $J R^{-T} \tilde{I}$ , one easily obtains (4.5).  $\square$

#### 4.2. Second inversion formula

Now we obtain below another formula for the inverse of a confluent polynomial-like matrix, by use of its  $\{J_{1/x}, I, I, C_Q(Q_n)\}$ -displacement representation. The following Lemma 4.3 see [2].

**Lemma 4.3.** Let  $Q$  be a polynomial system specified by (2.1), and  $C_Q(Q_n)$  be the confederate matrix as in (2.8) of  $Q_n(x)$  with respect to  $Q$ . Let  $\hat{Q}$  be the system of associated polynomials given by (3.1) and (3.2). Then the last row of the matrix  $\hat{Q}_k(C_Q(Q_n))$  is given by

$$e_n^T \cdot \hat{Q}_k(C_Q(Q_n)) = \alpha_n e_{n-k}^T. \quad (4.9)$$

**Lemma 4.4.** Let  $Q$  be as in (2.1), and  $C_Q(Q_n)$  be the confederate matrix of  $Q_n(x)$  with respect to  $Q$  and let  $G, B$  be given as in (4.0). If  $Q_n(x)$  is coprime with  $f(x) = \prod_{i=1}^t (x - x_i)^{n_i}$ , then the unique solution of the equation

$$\nabla_{\{J_{1/x}, I, I, C_Q(Q_n)\}}(R) = J_x R - R C_Q(Q_n) = G B \quad (4.10)$$

is given by

$$\begin{aligned} R &= \sum_{k=1}^{\alpha} \text{diag} \left( \frac{g_{10}^{(k)}}{Q_n(x_1)} \cdots \frac{g_{1,n_1-1}^{(k)} (n_1 - 1)!}{Q_n^{(n_1-1)}(x_1)} \cdots \frac{g_{t0}^{(k)}}{Q_n(x_t)} \cdots \frac{g_{t,n_t-1}^{(k)} (n_t - 1)!}{Q_n^{(n_t-1)}(x_t)} \right) \\ &\quad \times V_Q(x) \cdot \sum_{j=1}^n b_{k,n+1-j} \hat{Q}_{j-1}(C_Q(Q_n)). \end{aligned} \quad (4.11)$$

**Proof.** Note that the uniqueness of solution of Eq. (4.10) is obvious, since the spectrum of  $C_Q(Q_n)$  coincides with the zeros of  $Q_n(x)$  and therefore has no intersection with the spectrum of  $J_x$  by assumption. Substituting  $R$  given by (4.11) into (4.10), we have

$$\begin{aligned} & \nabla_{\{J_x, I, I, C_Q(Q_n)\}}(R) \\ &= \sum_{k=1}^{\alpha} \text{diag} \left( \frac{g_{10}^{(k)}}{Q_n(x_1)} \cdots \frac{g_{1,n_1-1}^{(k)} \cdot (n-1)!}{Q_n^{(n_1-1)}(x_1)} \cdots \frac{g_{t0}^{(k)}}{Q_n(x_t)} \cdots \frac{g_{t,n_t-1}^{(k)} \cdot (n_t-1)!}{Q_n^{(n_t-1)}(x_t)} \right) \\ & \quad \times [J_x V_Q(x) - V_Q(x) C_Q(Q_n(x))] \cdot \sum_{j=1}^n b_{k,n+1-j} Q_{j-1}(C_Q(Q_n)). \end{aligned}$$

Furthermore, using (2.9), we have

$$\begin{aligned} & \nabla_{\{J_x, I, I, C_Q(Q_n)\}}(R) \\ &= \sum_{k=1}^{\alpha} \text{diag} \left( \frac{g_{10}^{(k)}}{Q_n(x_1)} \cdots \frac{g_{1,n_1-1}^{(k)} \cdot (n_1-1)!}{Q_n^{(n_1-1)}(x_1)} \cdots \frac{g_{t0}^{(k)}}{Q_n(x_t)} \cdots \frac{g_{t,n_t-1}^{(k)} \cdot (n_t-1)!}{Q_n^{(n_t-1)}(x_t)} \right) \\ & \quad \times \begin{bmatrix} Q_n(x_1) \\ \vdots \\ \frac{Q_n^{(n_1-1)}(x_1)}{(n_1-1)!} \\ \vdots \\ Q_n(x_t) \\ \vdots \\ \frac{Q_n^{(n_t-1)}(x_t)}{(n_t-1)!} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\alpha_n} \end{bmatrix} \cdot \sum_{j=1}^n b_{k,n+1-j} \hat{Q}_{j-1}(C_Q(Q_n)) \\ &= \sum_{k=1}^{\alpha} g^{(k)} \cdot [b_{k1} \quad b_{k2} \quad \cdots \quad b_{kn}] = GB \end{aligned}$$

where we have used (4.9).  $\square$

**Theorem 4.5.** Let  $Q$  be the same as (2.1). Then

$$\text{rank } \nabla_{\{J_x, I, I, C_Q(Q_n)\}}(R) = \text{rank } \nabla_{\{J_x, I, I, C_{\hat{Q}}(Q_n)\}}(JR^{-T}\tilde{I}). \quad (4.12)$$

Moreover, if  $R$  is specified by its generator  $\{G, B\}$  on the right-hand side of

$$\nabla_{\{J_{1/x}, I, I, C_Q(Q_n)\}}(R) = J_{1/x}R - RC_Q(Q_n) = GB, \quad (4.13)$$

then

$$R^{-1} = \tilde{I} \cdot \sum_{k=1}^{\alpha} \left( \sum_{j=1}^n d_{kj} Q_{j-1} (C_{\hat{Q}}^T(Q_n)) \right) \cdot V_{\hat{Q}}^T \cdot \text{diag}(c_k) \cdot J, \quad (4.14)$$

where  $c_k \in C^n$  and  $d_{kj} \in C$  are determined from  $2\alpha$  linear systems of equations

$$\begin{aligned} [c_1, \dots, c_{\alpha}] &= \text{diag} \left( \frac{1}{Q_n(x_1)} \dots \frac{(n_1-1)!}{Q_n^{(n_1-1)}(x_1)} \dots \frac{1}{Q_n(x_t)} \dots \frac{(n_t-1)!}{Q_n^{(n_t-1)}(x_t)} \right), \\ &\quad \times J R^{-T} B^T \in C^{n \times \alpha} \\ [d_{kj}] &= G^T R^{-T} \in C^{\alpha \times n}. \end{aligned} \quad (4.15)$$

**Proof.** Multiplying (4.13) by  $\tilde{I}R^{-1}$  from the left and by  $R^{-1}$  from the right and then taking transpose, we obtain

$$J_x^T (R^{-T} \tilde{I}) - (R^{-T} \tilde{I}) C_{\hat{Q}}(Q_n) = (R^{-T} B^T) (G^T R^{-T} \tilde{I})$$

or equivalently

$$J_x (J R^{-T} \tilde{I}) - (J R^{-T} \tilde{I}) C_{\hat{Q}}(Q_n) = (J R^{-T} B^T) (G^T R^{-T} \tilde{I}). \quad (4.16)$$

Eq. (4.16) implies that equality (4.12) holds, and by writing (4.11) for the matrix  $J R^{-T} \tilde{I}$ , one easily obtains (4.14).  $\square$

#### 4.3. Special cases

As the special cases of Theorem 4.2, when  $Q$  stands for the power basis  $P$ , Chebyshev polynomials bases  $U$  and  $T$ , respectively, we get the inversion formulas of confluent Vandermonde-like and Chebyshev Vandermonde-like matrices, respectively, which are the generalizations of [24], [3] and [12].

When  $Q$  is equal to  $P$ , from (3.5) and (2.12) we have  $\hat{Q} = \hat{P} = P$  and  $W_{\hat{Q}} = W_P = Z_0$ , then Theorem 4.2 being applied to (2.17) we obtain

**Theorem 4.6.** Let  $P$  stand for the standard power basis and let  $R$  be a confluent Vandermonde-like matrix specified by its generator  $\{G, B\}$  on the right-hand side of

$$\nabla_{\{J_{1/x}, I, I, Z_0\}}(R) = J_{1/x} R - R Z_0 = G B. \quad (4.17)$$

Then

$$R^{-1} = \sum_{k=1}^{\alpha} H(d_k) \cdot (J V_P(x))^T \cdot L(c^{(k)}), \quad (4.18)$$

where  $c^{(k)}$  and  $d_k$  are determined from  $2\alpha$  linear systems of equations

$$\begin{aligned} [c^{(1)}, \dots, c^{(\alpha)}] &= J_x \cdot J R^{-T} B^T \in C^{n \times \alpha}, \\ \begin{bmatrix} d_1 \\ \vdots \\ d_\alpha \end{bmatrix} &= G^T R^{-T} \in C^{\alpha \times n}, \end{aligned} \quad (4.19)$$

and here and after  $H(d_k)$  denotes the upper triangular Hankel matrix with first row  $d_k$ .

Similarly, let Theorem 4.2 be applied to (2.18) and (2.19), and taking (3.6) and (3.7) into account, we have

**Theorem 4.7.** *Let  $U$  stand for the Chebyshev polynomials basis of the second kind and let  $R$  be specified by its generator  $\{G, B\}$  on the right-hand side of*

$$\nabla_{\{J_{1/x}, I, I, W\}}(R) = J_{1/x} R - R W = G B. \quad (4.20)$$

Then

$$R^{-1} = \sum_{k=1}^{\alpha} H(a_k) \cdot V_U^T(x) \cdot L(c^{(k)}), \quad (4.21)$$

where  $W = 2 \sum_{i=1}^{[n/2]} (-1)^{i-1} Z_0^{2i-1}$ ,  $c^{(k)}$  and  $a_k$  are determined by the  $2\alpha$  linear systems of equations

$$\begin{aligned} [c^{(1)} \quad \dots \quad c^{(\alpha)}] &= J_x \cdot J R^{-T} B^T \in C^{n \times \alpha}, \\ [\psi_1 \quad \dots \quad \psi_n] &= G^T R^{-T} \tilde{I} \in C^{\alpha \times n}, \end{aligned} \quad (4.22)$$

where  $\psi_i = \text{col}(\psi_{ki})_{k=1}^{\alpha}$  and  $a_k = \text{row}(a_{ki})_{i=1}^n$  with  $a_{k1} = \psi_{k1}$ ,  $a_{k2} = \psi_{k2}$ , and  $a_{ki} = \psi_{ki} + \psi_{k-2,i}$  for  $i = 3, 4, \dots, n$ .

**Theorem 4.8.** *Let  $T$  stand for the Chebyshev polynomials basis of the first kind and let  $R$  be specified by its generator  $\{G, B\}$ ,*

$$\nabla_{\{J_{1/x}, I, I, W\}}(R D_0) = J_{1/x}(R D_0) - (R D_0) W = G B. \quad (4.23)$$

Then

$$R^{-1} = \sum_{k=1}^{\alpha} 2 D_0 \cdot H(a_k) V_U^T(x) \cdot L(c^{(k)}), \quad (4.24)$$

where  $W$ ,  $c^{(k)}$  and  $\psi_i$  are the same as in Theorem 4.7,  $D_0 = \text{diag}(\frac{1}{2} \ 1 \ \dots \ 1)$  and  $a_k = \text{row}(a_{ki})_{i=1}^n$  with  $a_{ki} = 2\psi_{ki} + 4 \sum_{s=1}^{[\frac{n-1}{2}]} \psi_{k-2s,i}$ .



## 5. Fast block Gaussian elimination and generalized block Parker–Traub algorithm

### 5.1. Block Gaussian elimination

In deriving the formulas of confluent polynomial Vandermonde-like matrix  $R$  in Section 4 we have to solve linear systems with  $R$  as coefficient matrix (see (4.6), (4.19) and (4.22)) and  $R$  satisfies the Sylvester-type displacement equations (4.4), (4.17), (4.20) and (4.23). We point out that all these equations are the special cases of the following general Sylvester-type displacement equation:

$$\nabla_{[\Omega_1, A_1]}(R_1) = \Omega_1 R_1 - R_1 A_1 = G_1 B_1, \quad (5.1)$$

where  $\Omega_1, A_1$  are lower and upper triangular matrices with pairwise distinct main diagonal elements, respectively,  $G_1 \in C^{n \times \alpha}$ ,  $B_1 \in C^{\alpha \times n}$ . In [9] fast triangularization and fast Gaussian elimination algorithm with partial pivoting were given for the displacement equation of the form (5.1) (for details, see [2,9,14]), therefore, the algorithm in [9] is applied to the confluent polynomial Vandermonde-like matrix. For the confluent case, however, we may consider block Gaussian elimination for a confluent polynomial-like matrix. The following lemma is the block matrix case of [2, Lemma 6.1] (see [4,11,13]).

**Lemma 5.1.** *Let the matrix*

$$R_1 = \begin{bmatrix} D_1 & U_1 \\ L_1 & R_{22}^{(1)} \end{bmatrix} \quad (D_1 \in C^{n_1 \times n_1})$$

*satisfy the Sylvester displacement equation (5.1). If the (1,1) block  $D_1$  of  $R_1$  is non-singular, then the Schur complement  $R_2 = R_{22}^{(1)} - L_1 D_1^{-1} U_1$  satisfies the Sylvester displacement equation*

$$\Omega_2 R_2 - R_2 A_2 = G_2 B_2, \quad (5.2)$$

*where  $\Omega_2$  and  $A_2$  are obtained from*

$$\Omega_1 = \begin{bmatrix} W_1 & O \\ * & \Omega_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} A^{(1)} & * \\ O & A_2 \end{bmatrix}$$

*by deleting the first  $n_1$  rows and  $n_1$  columns, and*

$$\begin{bmatrix} O \\ G_2 \end{bmatrix} = G_1 - \begin{bmatrix} I_{n_1} \\ L_1 D_1^{-1} \end{bmatrix} G^{(1)}, \quad (5.3)$$

$$\begin{bmatrix} O & B_2 \end{bmatrix} = B_1 - B^{(1)} \begin{bmatrix} I_{n_1} & D_1^{-1} U_1 \end{bmatrix},$$

*where  $G^{(1)}$  and  $B^{(1)}$  are the first  $n_1$  rows of  $G_1$  and the first  $n_1$  columns of  $B_1$ , respectively.*

Let  $E_1 = [I_{n_1} \ 0 \ \cdots \ 0]^T \in C^{n \times n_1}$  and let us multiply by  $E_1$  from the right the Eq. (5.1) with  $\Omega_1 = J_{1/x}$  and  $A_1 = W_Q = N_Q M_Q^{-1}$ . Then one obtains the following recursive expression for the first  $n_1$  columns of  $R_1$ :

$$J_{1/x_1}(R_1 E_1) - R_1 W_Q E_1 = G_1 B^{(1)}$$

or

$$(R_1 E_1) - J_{x_1} R_1 W_Q E_1 = J_{x_1} G_1 B^{(1)}. \quad (5.4)$$

Similarly, multiplying by  $E_1^T$  from the left the Eq. (5.1), in which  $\Omega_1 = J_{1/x}$  and  $A_1 = W_Q$ , one obtains the recursive expression for the first  $n_1$  rows of  $R_1$ :

$$J_{1/x_1}(E_1^T R_1) - (E_1^T R_1) W_Q = G^{(1)}$$

or

$$(E_1^T R_1) - J_{x_1}(E_1^T R_1) W_Q = J_{x_1} G^{(1)} B_1, \quad (5.5)$$

where  $G^{(1)}$  is the first  $n_1$  rows of  $G_1$ .

Finally, by the upper triangular form of  $M_Q$ , we have

$$\Omega_2 = \text{diag}(J_{1/x_2} \ \cdots \ J_{1/x_t}), \quad A_2 = N_{Q,2} M_{Q,2}^{-1}, \quad (5.6)$$

where  $N_{Q,2}$  and  $M_{Q,2}$  are obtained by deleting the first  $n_1$  rows and  $n_1$  columns from  $N_Q$  and  $M_Q$ , respectively. Using these arguments, we shall write down an implementation of block Gaussian elimination for confluent polynomial Vandermonde-like matrices as follows:

1. Compute the first  $n_1$  columns of  $R_1$  via (5.4) recursively.
2. Compute the first  $n_1$  rows of  $R_1$  by solving the triangular linear system (5.5) by back substitution and recursively.
3. Write down the first  $n_1$  columns  $\begin{bmatrix} I_{n_1} \\ L_1 D_1^{-1} \end{bmatrix}$  of  $L$  and first  $n_1$  rows  $[D_1 \ U_1]$  of  $R$  in the  $LU$  decomposition of  $R_1$ .
4. Compute a generator  $\{G_2, B_2\}$  for the Schur complement  $R_2$  by using (5.3).

Finally, we point out that the overall complexity of the above algorithm is also  $O(m\alpha n^2)$  arithmetic operations as simple case (see, e.g., [2]), where the polynomials system  $Q$  in (2.1) satisfies  $m$ -term recurrence relations.

## 5.2. Generalized block Parker–Traub algorithm for inversion of confluent polynomial Vandermonde-like matrices

In this part we give the generalization of [2, Theorem 7.1] to the confluent polynomial Vandermonde matrices satisfying  $m$ -term recurrence relations, showing that

the complexity of inverting such a matrix is also  $O(m\alpha n^2)$  operations. The proofs are much the same way as [2, Theorem 7.1 and Lemma 7.2]. Here we omit.

**Theorem 5.2.** *Let  $Q$  be a system of polynomials satisfying  $m$ -term recurrence relations, and let matrices  $W_Q$  and  $J_{1/x}$  be given as before. Let the  $m$ -term confluent polynomial Vandermonde-like matrix  $R$  be given by its  $\{J_{1/x}, I, I, W_Q\}$ -generator  $G \in C^{n \times \alpha}$ ,  $B \in C^{\alpha \times n}$  on the right-hand side of*

$$\nabla_{\{J_{1/x}, I, I, W_Q\}}(R) = J_{1/x}R - RW_Q = GB.$$

*Then all  $n^2$  entries of  $R^{-1}$  can be computed in  $O(\alpha mn^2)$  operations.*

## 6. Transformation into confluent Cauchy-like matrices

Association with a pair of multiple nodes sequences

$$x = ([x_1, n_1], \dots, [x_t, n_t]) \quad \text{and} \quad y = ([y_1, m_1], \dots, [y_s, m_s]),$$

where  $x_i$  and  $y_j$  are all distinct for  $i = 1, \dots, t$ ,  $j = 1, \dots, s$ , we define a confluent Cauchy matrix

$$C(x, y) = [C_{ij}]_{i,j=1}^{t,s},$$

where  $C_{ij} = (C_{ij}^{kl})_{k=0, l=0}^{n_i-1, m_j-1}$  with

$$C_{ij}^{kl} = \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \left[ \frac{1}{x-y} \right]_{x=x_i}^{y=y_j} = \binom{k+l}{k} \frac{(-1)^k}{(x_i - y_j)^{k+l+1}}$$

(see, e.g., [1,25]).

The confluent Cauchy matrix satisfies the following displacement equation (see [17,19]):

$$\begin{aligned} \nabla_{\{J_x, I, I, J_y^T\}}(C(x, y)) &= J_x C(x, y) - C(x, y) J_y^T \\ &= \begin{bmatrix} e_{n_1} \\ e_{n_2} \\ \vdots \\ e_{n_t} \end{bmatrix} \begin{bmatrix} e_{m_1}^T & e_{m_2}^T & \cdots & e_{m_s}^T \end{bmatrix}, \end{aligned} \quad (6.1)$$

where  $J_x = \text{diag}(J_{x_i})_{i=1}^t$  and  $J_y = \text{diag}(J_{y_j})_{j=1}^s$  are the (lower triangular) Jordan canonical forms corresponding to  $x$  and  $y$ , respectively, and  $e_{n_i} = [1 \ 0 \ \cdots \ 0]^T \in C^{n_i}$  and  $e_{m_j} = [1 \ 0 \ \cdots \ 0]^T \in C^{m_j}$  are columns with first entry 1 and other entries 0.

Following [16], *confluent Cauchy-like or generalized confluent Cauchy* matrices can be defined as matrices with low  $\{J_x, I, I, J_y^T\}$ -displacement rank. For the

detailed discussion of generalized confluent Cauchy and Vandermonde (GCC and GCCV) matrices, one can refer to [19]. Here we only point out that since  $J_x, J_y^T$  are lower and upper triangular matrices, respectively, a linear system with confluent Cauchy-like matrix can be solved in  $O(n^2)$  operations. In the remained part of this section we give the result of transforming a confluent polynomial Vandermonde-like matrix into a confluent Cauchy-like matrix.

**Theorem 6.1.** *Let  $Q = \{Q_0(x), \dots, Q_{n-1}(x)\}$  be a system of polynomials satisfying (2.1), and let  $\Theta(x) = \prod_{j=1}^s (x - y_j)^{m_j}$ , where  $y_j$  are pairwise distinct. Let  $R$  be given by its  $\{J_x, I, I, C_Q(\Theta)\}$ -generator  $G \in C^{n \times \alpha}$ ,  $B \in C^{\alpha \times n}$ ,*

$$\nabla_{\{J_x, I, I, C_Q(\Theta)\}}(R) = J_x R - R C_Q(\Theta) = G B. \quad (6.2)$$

Then  $RV_Q(y)^{-1} J'$  is a confluent Cauchy-like matrix:

$$\begin{aligned} \nabla_{\{J_x, I, I, J_y^T\}}(RV_Q(y)^{-1} J') &= J_x [RV_Q(y)^{-1} J'] - [RV_Q(y)^{-1} J'] J_y^T \\ &= G \cdot [BV_Q(y)^{-1} J'], \end{aligned} \quad (6.3)$$

where  $J' = \text{diag}(J_j)_{j=1}^s$ ,  $J_j$  is the  $m_j \times m_j$  antiidentity matrix.

**Proof.** Note that in the confluent case the matrix  $C_Q(\Theta)$  is block diagonalized by  $V_Q(y) : C_Q(\Theta) = V_Q(y)^{-1} J_y V_Q(y)$  (see (2.10)), also  $J_y = J' J_y^T J' = J'^{-1} J_y^T J'^{-1}$ , therefore,  $C_Q(\Theta) = V_Q(y)^{-1} J' \cdot J_y^T \cdot J' V_Q(y)$ . Substituting the latter expression into (6.2) and then multiplying  $V_Q(y)^{-1} J'$  from the right, one obtains (6.3).  $\square$

## 7. Concluding remark

In this paper we have gotten the displacement structures, inversion formulas and fast algorithm for confluent polynomial Vandermonde-like matrices by use of displacement structure theory, but here we point out that all the ordinary Vandermonde matrices—including polynomial Vandermonde-like matrices, are defined with respect to a system  $Q$  of polynomials and a nodes sequence  $x = [(x_1, n_1), \dots, (x_t, n_t)]$ , where  $x_i$  are distinct complex numbers, or with respect to a system  $Q$  of polynomials and a *fundamental polynomial*  $f(x) = \prod_{i=1}^t (x - x_i)^{n_i}$ . In other words, the ordinary Vandermonde matrices are defined over complex field  $\mathbf{C}$  or an arbitrary algebraically closed field  $\mathbf{F}$ . In fact, the Vandermonde matrix has been generalized into any non-algebraically closed field case—*q-adic Vandermonde matrix* (see [29]) with respect to standard power basis  $P$  and a fundamental polynomial  $q(x) = \prod_{i=1}^t q_i(x)^{n_i}$ , where  $\deg q_i(x) = l_i (\geq 1)$  are arbitrary. The *q-adic Vandermonde matrix* was also used to reduce Bezout and Hankel matrices [18,23]. We note that the *q-adic Vandermonde matrix* has almost the same properties as the ordinary Vandermonde mat-

rix, such as the displacement structure and inversion formulas etc., which will be discussed in other papers.

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